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CALCULATION OF THE TEMPERATURE FIELDS OF SOLID BODIES OF BASE GEOMETRY WITH ARBITRARY BOUNDARY CONDITIONS

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A brief analysis of the method of calculation of the temperature fields of solid bodies of base geometry as applied to metallurgical production is made.

Introduction. In studying the stressed state of a solid body on the basis of the quasistatic theory of thermoelasticity, one must primarily obtain the so-called "load" function (temperature *T*) by independent solution of the corresponding boundary-value problem of heat conduction.

Study of the processes of heating and cooling of solid bodies is based on the theory of heat conduction, the modern advances of which are reported in numerous works, for example, [1–4].

The analytical method of investigation of these processes assumes the following basic steps: formulation of the problem and its solution, analysis of the data obtained, and numerical calculations of a specific object. All of them reflect different aspects of a single generalizing notion — a mathematical model of the process.

The formulation of the corresponding boundary-value problem involves selection of a mathematical model corresponding, to a certain extent, to the physical process under study. This model includes the differential heat-conduction equation, the conditions of heat exchange on the body's surface (boundary conditions), and the temperature state of a body before the beginning of the process under study (initial condition).

In the general case where the temperature is a function of three coordinates and time and the thermophysical characteristics can be assumed to be constant, the differential heat-conduction equation in a Cartesian coordinate system has the form

$$\lambda\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\omega=c\gamma\frac{\partial T}{\partial t}\,. \tag{1}$$

When the combining parameter m of the body's shape is introduced ($m = 0$ for a plate, $m = 1$ for a cylinder, and $m = 2$ for a sphere), the heat-conduction equation can be represented in generalized form [5, 6]:

$$\frac{a}{r^m}\frac{\partial}{\partial r}\left(r^m\frac{\partial T}{\partial r}\right)+\frac{\omega}{c\gamma}=\frac{\partial T}{\partial t}\,. \tag{2}$$

Expression (2) assumes the constancy of the parameters λ, c, γ, and a. If the thermophysical characteristics of the material are coordinate dependent, this expression is complicated but remains linear; it becomes nonlinear (nonlinearity of the first kind), when λ, c, and γ are temperature dependent.

Since the differential equation (2) has derivatives of second order with respect to coordinate and of first order with respect to time, its general solution will contain the corresponding number of arbitrary integration constants (or functions). One must determine these constants to obtain a unique solution. Therefore, in solving each specific problem

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of heat conduction, it is required that the corresponding uniqueness conditions, having the name of boundary (boundary and initial) conditions, be added to the differential equation of conductive heat transfer.

As the initial condition we prescribe the temperature of the body at the initial instant of time $t = 0$:

$$T(r, t)\big|_{t=0} = T(r, 0) = T_0(r) . \tag{3}$$

Establishment of the boundary conditions is a more complex process. One recognizes four kinds of boundary conditions of heat exchange.

Boundary conditions of the first kind prescribe the value of the temperature on the body's surface:

$$T(r, t)_{r=R} = T_s(t) . \tag{4}$$

Thus, $T_s(t)$ is the a priori known law of variation of the surface temperature of the body with time. The simplest case of condition (4)

$$T(r, t)_{r=R} = T_s = \text{const}$$

is possible either in high-intensity heat exchange, known as a "temperature shock", or in the case of artificial provision of a constant surface temperature throughout the process of heat transfer (thermostatic character).

Boundary conditions of the first kind are used in problems of heating of furnace linings, heat treatment of a metal, and hardening, heating, or cooling of bodies in liquid media.

With a boundary condition of the second kind, one prescribes the heat flux

$$-\lambda \frac{\partial T}{\partial r}\bigg|_{r=R} = q_s(t) . \tag{5}$$

on the body's surface.

A boundary condition of the third kind assumes that the relationship between the temperature and its normal derivative (heat flux) on the body's surface is known. If this relationship is linear, the boundary condition, in accordance with the Newton–Richmann law of convective heat exchange, has the form

$$\lambda \frac{\partial T}{\partial r}\bigg|_{r=R} = \alpha\,[T_m(t) - T_s(t)] . \tag{6}$$

In individual cases, for $\alpha \to \infty$ and $\alpha = 0$ the boundary condition of the third kind (6) becomes boundary conditions of the first and second kind.

If the relationship between the temperature and the heat flux on the body's surface is nonlinear, such a boundary condition makes the entire boundary-value problem of heat conduction nonlinear (nonlinearity of the second kind). The most widespread boundary condition describes, according to the Stefan–Boltzmann law, radiation heat exchange:

$$\lambda \frac{\partial T}{\partial r}\bigg|_{r=R} = \sigma_{app}\,[T_m^4(t) - T_s^4(t)] . \tag{7}$$

Boundary conditions of the fourth kind reflect the so-called conjugate heat exchange, which assumes the existence of temperature and heat balances on the combined surface of contacting media. This is expressed by the following conditions:

$$T_1(r, t)\big|_{c.s} = T_2(r, t)\big|_{c.s} , \quad \lambda_1 \frac{\partial T_1}{\partial r}\bigg|_{c.s} = \lambda_2 \frac{\partial T_2}{\partial r}\bigg|_{c.s} .$$

These conditions are most frequently employed in calculations of the processes of heat conduction in multilayer walls. Sometimes, they are used in investigating convective heat exchange between a solid body and a liquid since the Newton–Richmann law does not necessarily give reliable results, which is explained by the difficulty and arbitrariness of the determination of α.

Use of the Method of Equivalent Sources. We consider the application of the method named the "method of equivalent sources" in the literature [6–8] to solution of the problems described above.

First of all, we introduce the relative excess function of temperature $\theta(\rho, \mathrm{Fo}) = T(\rho - \mathrm{Fo}) - T_0/(T^* - T_0)$, the dimensionless independent variables $\rho = r/R$ and $\tau = \mathrm{Fo} = at/R^2$, and the number $\mathrm{Po} = \omega R^2/[\lambda(T^* - T_0)]$. Here T^* = const is a certain characteristic temperature, for example, the temperature of a burning medium T_m = const or the body's surface T_s = const.

For convenience of further calculations, we locate the origin of coordinates on the body's surface ($\xi = 1 - \rho$). In such a case, for Po = 0 the heat-conduction equation (2) can be represented as

$$\frac{1}{(1-\xi)^m}\frac{\partial}{\partial \xi}\left[(1-\xi)^m \frac{\partial \theta}{\partial \xi}\right] = \frac{\partial \theta}{\partial \tau}\,, \quad \theta(\xi, 0) = 0\,. \tag{8}$$

Then the *boundary condition of the first kind* acquires the form

$$\theta(\xi, \tau)\big|_{\xi=0} = \theta_s(\tau) = \theta_m(\tau)\,, \tag{9}$$

where $\theta(\xi, \tau) = T(\xi, \tau) - T_0/T^*$ and $\theta_m(\tau) = T_m(\tau) - T_0/T^*$, T^* being a certain (scale) temperature, for example, T_0, $T_m(0)$, T_m^{max}, $T_m^{max} - T_0$, $T_m(0) - T_0$, and others.

The essence of the method of equivalent sources [6–8] is that it combines the method of successive approximations and integral methods [9, 10]. It does not necessitate a priori approximation of the temperature field and yet is close to the well-known method of averaging of functional corrections of Yu. D. Sokolov [11].

The scheme of its employment is as follows. We apply the method of successive approximations to the initial equation (8), writing it in the form

$$\frac{1}{(1-\xi)^m}\frac{\partial}{\partial \xi}\left[(1-\xi)^m \frac{\partial \theta_{n+1}}{\partial \xi}\right] = \frac{\partial \theta_n}{\partial \tau}\,. \tag{10}$$

We introduce into Eq. (10) the additional term $f_{n+1}(\tau)$, which is equivalent to the residual occurring in Eq. (8) after the substitution of the nth approximation into the right-hand side and the $(n+1)$th approximation into the left-hand side:

$$\frac{1}{(1-\xi)^m}\frac{\partial}{\partial \xi}\left[(1-\xi)^m \frac{\partial \theta_{n+1}}{\partial \xi}\right] + f_{n+1}(\tau) = \frac{\partial \theta_n}{\partial \tau}\,. \tag{11}$$

Here the function $f_{n+1}(\tau)$ acts as the "source."

Let it be necessary that the $(n+1)$th approximation integrally satisfy the initial equation (8):

$$\int_0^l \left\{\frac{1}{(1-\xi)^m}\frac{\partial}{\partial \xi}\left[(1-\xi)^m \frac{\partial \theta_{n+1}}{\partial \xi}\right] - \frac{\partial \theta_{n+1}}{\partial \tau}\right\} d\xi = 0\,. \tag{12}$$

Carrying out functional integration of (11) for ξ going from 0 to ξ, we have

$$\int_0^l \left\{\frac{1}{(1-\xi)^m}\frac{\partial}{\partial \xi}\left[(1-\xi)^m \frac{\partial \theta_{n+1}}{\partial \xi}\right] - \frac{\partial \theta_n}{\partial \tau}\right\} d\xi + f_{n+1}(\tau)\, l(\tau) = 0\,. \tag{13}$$

The difference between (12) and (13) leads to the expression

$$f_{n+1}(\tau) = \frac{1}{l}\int_0^l \left\{\frac{\partial\theta_n}{\partial\tau} - \frac{\partial\theta_{n+1}}{\partial\tau}\right\} d\xi\,, \tag{14}$$

which is an integral condition for determination of the unknown function $f_{n+1}(\tau)$. Here $l(\tau)$ is a certain size of the heated layer; a distinction needs to be drawn between the *inertial step* $(0 \le \tau \le \tau_0,\ 0 \le \xi \le l(\tau))$ in which heating follows the law of heating of a half-space, i.e., when the condition at infinity $\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=\infty} = 0$ is replaced by the condition $\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=l} = 0$, and the *ordered step* $(\tau_0 \le \tau \le \infty,\ 0 \le \xi \le 1)$.

Thus, if the nth approximation of the problem in question is known, the $(n+1)$th approximation of θ_{n+1} is determined by Eq. (11), the boundary conditions, and the integral expression (14).

Next, integrating (11) doubly with respect to ξ with allowance for the boundary conditions, we obtain the expression for θ_{ij}. Substituting it into the integral condition (14), we arrive at the differential equation for $f_{n+1}(\tau)$. The process of determination of θ_{n+1} ends with solution of this equation.

Let us consider a few examples of the employment of the method presented.

For problem (8), (9), in the inertial step we have

$$\theta_{11}(\xi, \tau) = \theta_m(\tau)\left[1 - \frac{\xi}{l(\tau)}\right]^2, \tag{15}$$

$$\frac{d}{d\tau}\left[\theta_m(\tau)\, l(\tau)\right] = 6\,(1 + m)\,\frac{\theta_m(\tau)}{l(\tau)}\,. \tag{16}$$

After the integration of Eq. (16) with the initial condition $l(0) = 0$, by substitution of $\eta = \xi/(2\sqrt{\tau})$ we obtain the formulas

$$l(\tau) = \sqrt{\frac{12\,(1+m)}{\theta_m^2(\tau)}\int_0^\tau \theta_m^2(\eta)\, d\eta}\,, \tag{17}$$

$$\int_0^{\tau_0} \theta_m^2(\tau)\, d\tau = \frac{\theta_m^2(\tau_0)}{12\,(1+m)}\,, \tag{18}$$

determining the depth $l(\tau)$ of heating and the duration τ_0 of existence of the inertial step in each specific case of prescription of the function $\theta_m(\tau)$.

We note that the employment of another variant of the method yields

$$\theta_{11}(\xi, \tau) = \theta_m(\tau)\,\frac{(1+m)(1-l)^2 - 2\,(1-l)^{1+m}(1-\xi)^{1-m} + (1-m)(1-\xi)^2}{(1-m) + (1+m)(1-l)^2 - 2\,(1-l)^{1+m}} \tag{19}$$

or separately for the plate ($m = 0$) (see (15)), the cylinder ($m = 1$) (after the expansion of the indeterminacy of the form 0:0)

$$\theta_{11}(\xi, \tau) = \theta_m(\tau)\,\frac{(1-\xi)^2 - (1-l)^2 + 2\,(1-l)^2 \ln\dfrac{1-l}{1-\xi}}{1 - (1-l)^2 + 2\,(1-l)^2 \ln(1-l)}$$

and the sphere ($m = 2$)

$$\theta_{11}(\xi, \tau) = \theta_m(\tau) \frac{(1-\xi)^2 - (1-l)^2 + 2(1-l)^2 \dfrac{l-\xi}{1-\xi}}{1-(1-l)^2 - 2(1-l)^2 l} .$$

The solution of (19) is somewhat more complex than that of (15) but it more accurately describes the temperature field in the process of heating of the cylinder and the sphere, which can become substantial in the case of a considerable thermal mass of the bodies.

In the ordered step ($\tau_0 \le \tau$), the temperature function sought has the form

$$\theta_{21}(\xi, \tau) = \theta_m(\tau) - \Delta\theta_{21}(\tau)\,[1-(1-\xi)^2] , \tag{20}$$

where we have introduced the temperature-difference function

$$\Delta\theta_{21}(\tau) = \theta_{2s}(\tau) - \theta_{2c}(\tau) = \left\{\theta_m(0) + 1.5 \int_{\tau_0}^{\tau} \theta_m(\eta) \exp[3(1+m)(\eta-\tau_0)]\,d\eta\right\} \exp[-3(1+m)(\tau-\tau_0)] . \tag{21}$$

Thus, we have obtained the approximate solution of problem (15) for general prescription of an arbitrary, if only piecewise continuous and differentiable function $\theta_m(\tau)$.

Examples illustrating a sufficient convergence of the approximate (21) and exact [3] solutions are given in [6, 12]. We consider some of them.

The surface temperature is constant, $\theta_s(\tau) = \theta_m = \text{const}$, and $T^* = T_m - T_0$. This simplest case has practical use in investigating the temperature state of bodies in a liquid medium, in holding a metal in continuous furnaces or oil hardening, etc.

The solution of (15), (17), (18), and (21) is simplified to the form

$$\theta_{11}(\xi, \tau) = \left[1 - \frac{\xi}{l(\tau)}\right]^2 , \quad 0 \le \xi \le l(\tau) , \quad 0 \le \tau \le \tau_0 ;$$

$$\theta_{21}(\xi, \tau) = 1 - \Delta\theta_{21}(\tau)\,[1-(1-\xi)^2] , \quad 0 \le \xi \le l , \quad \tau \ge \tau_0 ;$$

$$l(\tau) = 2\sqrt{3(1+m)\,\tau} , \quad \tau_0 = \frac{1}{12(1+m)} , \quad \Delta\theta_{21}(\tau) = \exp[-3(1+m)(\tau-\tau_0)] .$$

The surface temperature is a linear function:

$$T_m(t) = T_0 + V_T t , \quad \theta_m(\tau) = \text{Pd}\,\tau , \quad \text{Pd} = \frac{V_T R^2}{a T^*} .$$

This law is employed, for example, in investigation and calculations of the thermal processes of heating of a furnace lining or in heat treatment of a metal with a required rate.

If, following [3], we introduce a new temperature function

$$\bar{\theta}(\xi, \tau) = \frac{\theta_m(\xi, \tau)}{\text{Pd}\,\tau}$$

and determine the temperature difference (21)

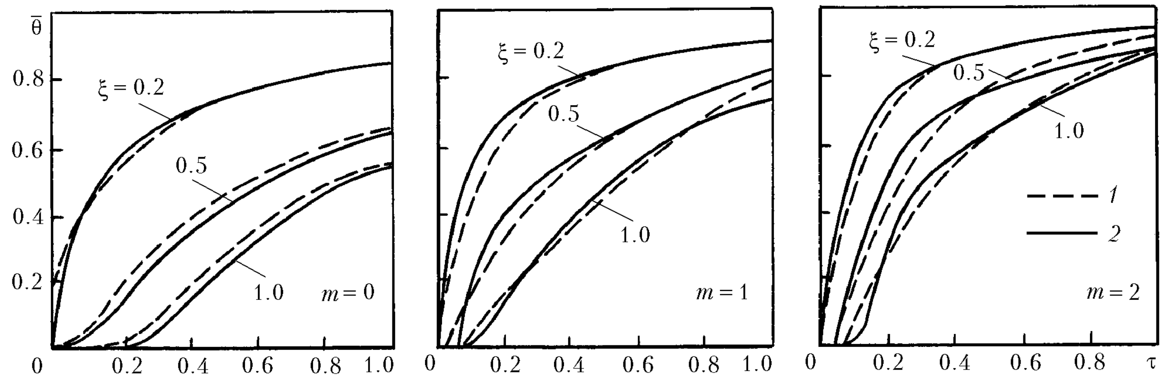


Fig. 1. Change in the temperature function $\bar{\theta}(\xi, \tau)$ at different points ξ of the cross section of a plate (m = 0), a cylinder (m = 1), and a sphere (m = 2) according to the exact (1) and approximate (2) solutions of the problem with boundary conditions of the first kind.

$$\Delta\bar{\theta}(\tau) = \frac{1 - [1 - 2(1+m)\tau_0] \exp[-3(1+m)(\tau - \tau_0)]}{2(1+m)\tau}, \tag{22}$$

from expressions (15) and (20) we obtain

$$\bar{\theta}_{11}(\xi, \tau) = \left[1 - \frac{\xi}{l(\tau)}\right]^2, \quad 0 \le \tau \le \tau_0, \quad 0 \le \xi \le l(\tau);$$

$$\bar{\theta}_{21}(\xi, \tau) = 1 - \Delta\bar{\theta}_2(\tau)\,[1 - (1 - \xi)^2], \quad \tau \ge \tau_0, \quad 0 \le \xi \le l; \tag{23}$$

$$l(\tau) = 2\sqrt{(1+m)\tau}, \quad \tau_0 = \frac{1}{4(1+m)}.$$

The comparison of the exact [3] and approximate (22), (23) solutions in Fig. 1 shows a fairly high correspondence of them.

We recall that a *boundary condition of the second kind* assumes the specific heat flux on the body's surface as a function of the coordinates and the time t in the general case and only of the time in one-dimensional bodies, i.e., $q_s(t)$, to be a priori known (see (5)):

$$\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=0} = -\,\mathrm{Ki}(\tau). \tag{24}$$

Rather widespread is a particular case of the boundary condition of the second kind for q_s = const:

$$\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=0} = -\,\mathrm{Ki} = \mathrm{const}.$$

Such boundary conditions are taken in investigating the processes of heating of a metal in soaking pits, chamber furnaces, and other heaters.

In the inertial step ($0 \le \tau \le \tau_0$), for condition (24) we have

$$\theta_{11}(\xi, \tau) = \frac{1}{2}\mathrm{Ki}(\tau)\, l(\tau) \left[1 - \frac{\xi}{l(\tau)}\right]^2, \tag{25}$$

$$l(\tau) = \sqrt{\frac{6(1+m)}{\mathrm{Ki}(\tau)} \int_0^{\tau} \mathrm{Ki}(\tau)\, d\tau}\ , \tag{26}$$

$$\int_0^{\tau_0} \mathrm{Ki}(\tau)\, d\tau = \frac{\mathrm{Ki}(\tau_0)}{6(1+m)}\ . \tag{27}$$

Expressions (25) and (26) determine the law of advance of the front $l(\tau)$ and the duration τ_0 of this step. If Ki = const, expressions (26) and (27) yield the formulas $l(\tau) = \sqrt{6(1+m)\tau}$ and $\tau_0 = [6(1+m)]^{-1}$. As $q_s(t) = V_q t$ and $\mathrm{Ki}(\tau) = \mathrm{Pk}\tau$, $\mathrm{Pk} = V_q R^3/(\lambda a T^*)$, change linearly, we find $l(\tau) = \sqrt{3(1+m)\tau}$ and $\tau_0 = 1/3(1+m)$.

At the instant $\tau = \tau_0$ of completion of the inertial step, the temperature field is described by function (25), in which we should set $l(\tau_0) = 1$:

$$\theta_{11}(\xi, \tau_0) = \frac{1}{2} \mathrm{Ki}(\tau_0)(1-\xi)^2\ .$$

For the ordered step $(\tau \geq \tau_0)$, the solution of

$$\theta_{21}(\xi, \tau) = \frac{-f_{21}(\tau)(1-\xi^2)}{2(1+m)} + B_{21}(\tau)$$

with account for (24) acquires the form [6, 12]

$$\theta_{21}(\xi, \tau) = \frac{\mathrm{Ki}(\tau)(1-\xi)^2}{2} + B_{21}(\tau)\ .$$

The function $B_{21}(\tau) = \theta_{2c}(\tau)$ is determined by the integral condition

$$f_{21}(\tau) = -\int_0^1 \frac{\partial \theta_{21}}{\partial \tau} d\xi = -\frac{d}{d\tau} \int_0^1 \theta_{21}(\xi, \tau)\, d\xi :$$

$$B_{21}(\tau) = \theta_{2c}(\tau) = (1+m) \int_{\tau_0}^{\tau} \mathrm{Ki}(\eta)\, d\eta - \frac{\mathrm{Ki}(\tau) - \mathrm{Ki}(\tau_0)}{6}\ . \tag{28}$$

Then we can write

$$\theta_{21}(\xi, \tau) = \theta_{2c}(\tau) + \Delta\theta_{21}(\tau)(1-\xi)^2\ , \tag{29}$$

where $\theta_{2c}(\tau)$ is determined from (28) and the temperature difference is

$$\Delta\theta_{21}(\tau) = \theta_{2s}(\tau) - \theta_{2c}(\tau) = \frac{\mathrm{Ki}(\tau)}{2}\ . \tag{30}$$

Thus, we have obtained the approximate solution of the problem with boundary condition (24) for an arbitrary law of variation of the surface specific heat flux $q_s(t)$.

In the particular case for $q_s(t)$ = const we have

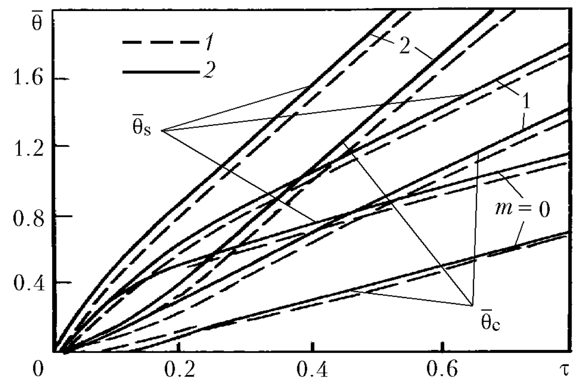


Fig. 2. Time change in the temperature function $\bar{\theta} = \theta/\mathrm{Ki}$ of the surface $\bar{\theta}_s$ and center $\bar{\theta}_c$ of the cross section of a plate (m = 0), a cylinder (m = 1), and a sphere (m = 2) according to the exact [3] (1) and approximate (31), (32) (2) solutions.

$$\theta_{21}(\xi, \tau) = \mathrm{Ki}\,\frac{(1-\xi)^2 + 2(1+m)(\tau - \tau_0)}{2}, \tag{31}$$

and for $q_s(t) = V_q t$ and $\mathrm{Ki}(\tau) = \mathrm{Pk}(\tau)$, $\mathrm{Pk} = V_q R^3/[\lambda a(T_m - T)]$, we have

$$\theta_{21}(\xi, \tau) = \mathrm{Pk}\,\tau\,\frac{(1-\xi)^2 - \dfrac{1 - \dfrac{\tau_0}{\tau}}{3} + (1+m)\,\tau\left(1 - \dfrac{\tau_0^2}{\tau^2}\right)}{2}.$$

At the same time, we obtain

$$\theta_{11}(\xi, \tau) = \mathrm{Ki}\,\frac{(1+m)(1-l)^2 - 2(1-l)^{1+m}(1-\xi)^{1-m} + (1+m)(1-\xi)^2}{2(1-m)[1-(1-l)^{1+m}]}, \tag{32}$$

where

$$l(\tau) = \sqrt{6\tau}, \quad \tau_0 = \frac{1}{6} \quad (m = 0)$$

$$l^2(\tau) - 4l(\tau) - 8\ln\left[1 - \frac{l(\tau)}{2}\right] = 24\tau,$$

$$\tau_0 = \frac{8\ln 2 - 3}{24} \quad (m = 1);$$

$$\frac{2\pi}{\sqrt{3}}\left\{1 - \frac{3}{\pi}\left[\arctan\left(\sqrt{3} - \frac{2l}{\sqrt{3}}\right)\right]\right\} - 2l - \ln\left(1 - l + \frac{l^2}{3}\right) = 12\tau,$$

$$\tau_0 = \frac{\pi + \sqrt{3}\ln 3 - 2\sqrt{3}}{12\sqrt{3}} \quad (m = 2).$$

The convergence of the exact and approximate solutions is clearly illustrated by Fig. 2.

One more case of practical importance is considered in [6, 12]. As is well known, most of the exact solutions of boundary-value heat-conduction problems in the expressions of specific heat fluxes contain exponential time functions. Setting

$$\mathrm{Ki}\,(\tau) = \mathrm{Ki}\,(0)\exp\,(-\rho\tau)\,, \tag{33}$$

in the general solution of (25)–(30), we determine [6, 9]

$$\theta_{11}\,(\xi,\tau) = \frac{\mathrm{Ki}\,(0)}{2}\left[1 - \frac{\xi}{l\,(\tau)}\right]^2 \exp\,(-\rho\tau)\,, \tag{34}$$

$$\theta_{21}\,(\xi,\tau) = \frac{\mathrm{Ki}}{2}\left[(1-\xi)^2 + \frac{2\,(1+m)}{\rho}\left\{\exp\,(\rho\tau) - \exp\,(-\rho\tau)\right\}\right]\exp\,(-\rho\tau)\,, \tag{35}$$

$$l^2\,(\tau) = \sqrt{\frac{6\,(1+m)}{\rho}}\;[\exp\,(\rho\tau) - 1]\,,\quad \tau_0 = \frac{1}{\rho}\ln\left[1 + \frac{6\,(1+m)}{\rho}\right]. \tag{36}$$

The exact solution of problem (24), (33), obtained in [13], has a rather complex form, whereas the approximate solution of (34)–(36) is much simpler and more practical. As far as the exactness is considered, the plots in Fig. 3 show a fairly high convergence of the approximations obtained.

Boundary conditions of the third kind assume that the nonstationary temperature $T_m(t)$ of the heating medium and the law of heat exchange between it and the body's surface are known in the general case. Depending on this law, the boundary conditions can be linear or nonlinear.

If the heat exchange between the body's surface and the medium surrounding it follows the Newton–Richmann law (6) with a nonstationary heat-transfer coefficient $\alpha(t)$

$$\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=0} = -\,\mathrm{Bi}\,(\tau)\,[\theta_m\,(\tau) - \theta_s\,(\tau)]\,, \tag{37}$$

this expression, in the theory of heat conduction, is called a "boundary condition of the third kind," where $\mathrm{Bi}(\tau) = \alpha(\tau)R/\lambda$ is the nonstationary analog of the Biot number.

Condition (37) is linear and describes convective heat exchange occurring in low-temperature furnaces or zones, soaking pits and liquid media, in cooling of products by air, water, etc.

In the inertial step $(0 \le \tau \le \tau_0)$, substituting the function

$$\theta_{11}\,(\xi,\tau) = -\frac{1}{2} f_{11}\,(\tau)\, l^2\,(\tau)\left[1 - \frac{\xi}{l\,(\tau)}\right]^2$$

into boundary condition (37), we obtain a relationship between $f_{11}(\tau)$ and $l(\tau)$:

$$f_{11}\,(\tau) = -\frac{2\,\mathrm{Bi}\,(\tau)\,\theta_m\,(\tau)}{l\,(\tau)\,[2 + \mathrm{Bi}\,(\tau)\,l\,(\tau)]}\,.$$

Thereafter, the expression $\theta_{11}(\xi,\tau)$ and the equation $\frac{d}{d\tau}[f_{11}(\tau)l^3(\tau)] = 6(1+m)f_{11}(\tau)l(\tau)$ are written as follows:

$$\theta_{11}\,(\xi,\tau) = \frac{\mathrm{Bi}\,(\tau)\,\theta_m\,(\tau)\,l\,(\tau)}{2 + \mathrm{Bi}\,(\tau)\,l\,(\tau)}\left[1 - \frac{\xi}{l\,(\tau)}\right]^2 = \Delta\theta_{11}\,(\tau)\left[1 - \frac{\xi}{l\,(\tau)}\right]^2, \tag{38}$$

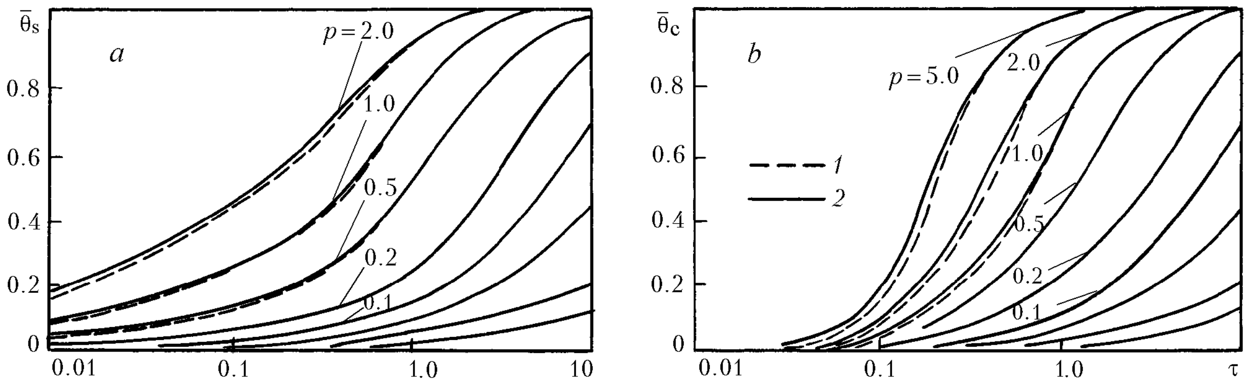


Fig. 3. Change in the temperature function $\bar{\theta} = \rho\theta/[2\text{Ki}(0)]$ of the surface (a) and center (b) of the cross section of a cylinder according to the exact [13] (1) and approximate (34)–(36) (2) solutions of problem (24), (33).

$$\frac{d}{d\tau}\left[\frac{\text{Bi}(\tau)\,\theta_m(\tau)\,l^2(\tau)}{2+\text{Bi}(\tau)\,l(\tau)}\right]=6\,(1+m)\,\frac{\text{Bi}(\tau)\,\theta_m(\tau)}{2+\text{Bi}(\tau)\,l(\tau)}\,. \tag{39}$$

We consider certain particular cases [6, 12] where Eq. (39) allows solutions in general form. First of all, when $\theta_m(\tau)$ = const it has the exact solution which is determined by the transcendental expression

$$l^2(\tau)+\frac{4l(\tau)}{\text{Bi}(\tau)}-\frac{8}{\text{Bi}^2(\tau)}\ln\left[1+\frac{\text{Bi}(\tau)\,l(\tau)}{2}\right]=12\,(1+m)\,\tau\,. \tag{40}$$

Furthermore, we can give ([6, 12]) two more approximate solutions of Eq. (39) for small and large Bi:

$$l(\tau)=\sqrt{\frac{6\,(1+m)}{\text{Bi}(\tau)\,\theta_m(\tau)}\int_0^\tau \text{Bi}(\eta)\,\theta_m(\eta)\,d\eta}\,,\quad \text{Bi}(\tau)\ll 2\,;$$

$$l(\tau)=\sqrt{\frac{12\,(1+m)}{\theta_m^2(\tau)}\int_0^\tau \theta_m^2(\eta)\,d\eta}\,,\quad \text{Bi}(\tau)\,l(\tau)\gg 2\,.$$

For moderate values of the Bi (τ) number we must solve Eq. (39) for each specific case of prescription of the functions $\theta_m(\tau)$ and Bi (τ).

The time τ_0 of completion of the inertial step is determined when $l(\tau_0) = 1$:

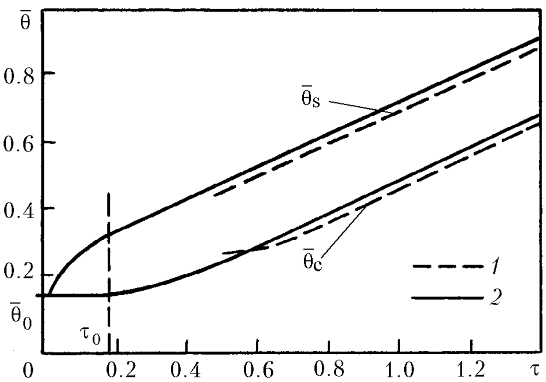
for Bi << 2

$$\int_0^{\tau_0}\text{Bi}(\eta)\,\theta_m(\eta)\,d\eta=\frac{\text{Bi}(\tau_0)\,\theta_m(\tau_0)}{6\,(1+m)}\,;$$

for Bi >> 2

$$\int_0^{\tau}\theta_m^2(\eta)\,d\eta=\frac{\theta_m^2(\tau_0)}{12\,(1+m)}\,;$$

for $\theta_m(\tau)$ = const



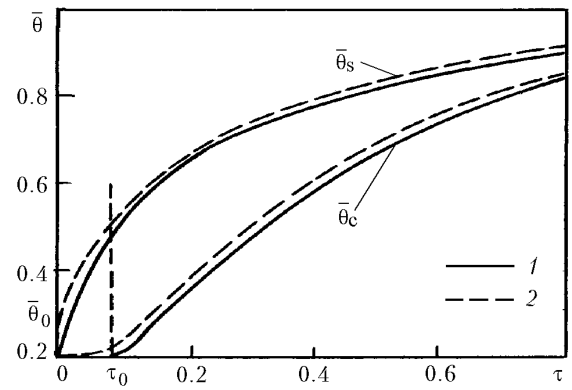


Fig. 4. Change in the temperature function $\bar{\theta}(\xi, \tau) = T(r, t)/[T_m(0)]$ of the surface θ_s and center θ_c of the cross section of a plate according to the solutions of the numerical method [14] (1) and (38), (42) (2) of the problem with boundary conditions of the third kind (37).

Fig. 5. Change in the temperature function $\bar{\theta}(\xi, \tau) = T(r, t)/[T_m(0)]$ of the surface θ_s and center θ_c of the cross section of a cylinder for the data (43) (1) and according to the nomograms of [15] (2).

$$\tau_0 = \frac{1}{12\,(1+m)}\left\{1 + \frac{4}{\mathrm{Bi}\,(\tau_0)} - \frac{8}{\mathrm{Bi}^2\,(\tau_0)}\ln\left[1 + \frac{\mathrm{Bi}\,(\tau_0)}{2}\right]\right\}.$$

Setting $\tau = \tau_0$ and $l(\tau_0) = 1$ in (38), we obtain the function of temperature at the end of the inertial step:

$$\theta_{11}\,(\xi, \tau_0) = \frac{\theta_m\,(\tau_0)\,\mathrm{Bi}\,(\tau_0)}{2 + \mathrm{Bi}\,(\tau_0)}\,(1-\xi)^2\,.$$

In the ordered step ($\tau \geq \tau_0$), after the employment of condition (37), the general approximate solution acquires the form

$$\theta_{21}\,(\xi, \tau) = \theta_m\,(\tau) + \frac{f_{21}\,(\tau)}{2\,(1+m)}\left[\frac{2 + \mathrm{Bi}\,(\tau)}{\mathrm{Bi}\,(\tau)} - (1-\xi)^2\right]. \tag{41}$$

Substituting function (41) into the integral condition $f_{21}(\tau) = -\int_0^l \frac{\partial\theta_{21}}{\partial\tau}d\xi$, we obtain

$$\theta_{21}\,(\xi, \tau) = \theta_m\,(\tau) - \Delta\theta_{21}\,(\tau)\left[\frac{2 + \mathrm{Bi}\,(\tau)}{\mathrm{Bi}\,(\tau)} - (1-\xi)^2\right], \tag{42}$$

where

$$\Delta\theta_{21}\,(\tau) = -\frac{f_{21}\,(\tau)}{2\,(1+m)} = \left\{\frac{\mathrm{Bi}\,(\tau_0)\,\theta_m\,(\tau_0)}{2 + \mathrm{Bi}\,(\tau_0)} + 1.5\int_{\tau_0}^{\tau} g\,(\tau)\exp\left[\int_{\tau_0}^{\tau} h\,(\tau)\,d\tau\right]d\tau\right\}\exp\left[-\int_{\tau_0}^{\tau} h\,(\tau)\,d\tau\right];$$

$$g\,(\tau) = \frac{\mathrm{Bi}\,(\tau)\,\theta_m\,(\tau)}{3 + \mathrm{Bi}\,(\tau)}; \quad h\,(\tau) = 3\,\frac{(1+m)\,\mathrm{Bi}^2\,(\tau) - \mathrm{Bi}\,(\tau)}{[3 + \mathrm{Bi}\,(\tau)]\,\mathrm{Bi}\,(\tau)}\,.$$

Thus, the problem formulated has been solved in general form for arbitrary if only piecewise continuous and differentiable functions $\mathrm{Bi}(\tau)$ and $\theta_m(\tau)$. For example, Fig. 4 [6, 12] gives changes in the temperature function $\bar{\theta}(\xi, \tau)$

$= T(\xi, \tau)/[T_m(0)]$ of the surface $\bar{\theta}_s$ and the median plane $\bar{\theta}_c$ of a plate ($m = 0$), plotted for the following data of convective heat exchange: $\text{Bi}(\tau) = \text{Bi}(0) \exp \tau$, $\text{Bi}(0) = 0.5$, $\theta_m(\tau) = 1 + \text{Pd}\,\tau$, $\text{Pd} = 0.075$, and $\theta(0) = 0.15$.

We give one more example. In [6, 12], we have calculated, from the solution (38)–(42), the relative (T/T_m) temperatures of the surface θ_s and center θ_c of the cross section of a cylinder ($m = 1$) for the following data:

$$\text{Bi}(\tau) = \text{Bi}(0) + b\tau\,, \quad \bar{\theta}(0) = 0.2\,, \quad \text{Bi}(0) = 1.5\,, \quad b = 0.45\,. \tag{43}$$

The results of the calculations are given in Fig. 5.

Let us consider the case of convective heating (or cooling) at a constant coefficient of heat exchange (α = const) and a nonstationary temperature $T_m(t)$ of the heating medium. Setting Bi (τ) = Bi = const in the general solution (38)–(42), we have [6, 12]

$$\theta_1(\xi, \tau) = \begin{cases} \dfrac{\theta_m(\tau)\,\text{Bi}\,l(\tau)}{2 + \text{Bi}\,l(\tau)}\left[1 - \dfrac{\xi}{l(\tau)}\right]^2, & 0 \le \xi \le l(\tau)\,, \quad 0 \le \tau \le \tau_0\,; \\ \theta_m(\tau) - \Delta\theta_{21}(\tau)\left[\dfrac{2 + \text{Bi}}{\text{Bi}} - (1 - \xi)^2\right], & 0 \le \xi \le 1\,, \quad \tau_0 \le \tau < \infty\,, \end{cases}$$

where

$$\Delta\theta_{21}(\tau) = \left\{\frac{\theta_m(\tau_0)\,\text{Bi}}{2 + \text{Bi}} + \frac{1.5\,\text{Bi}}{3 + \text{Bi}} \int_{\tau_0}^{\tau} \theta_m(\tau) \exp\left[\mu_m(\tau - \tau_0)\right] d\tau\right\} \exp\left[-\mu_m(\tau - \tau_0)\right];$$

$$\mu_m = \frac{3(1 + m)\,\text{Bi}}{3 + \text{Bi}}\,.$$

The thickness $l(\tau)$ of the heated layer for moderate values of Bi is determined by the solution of the differential equation (39) or the above formulas for small and large Bi.

Despite their obvious simplicity, the approximate solutions of the problems of heat conduction with nonstationary linear boundary conditions obtained have an exactness sufficient for everyday practice and can be applied to thermomechanical calculations.

NOTATION

$a = \lambda/c\gamma$, thermal diffusivity, m^2/sec; Bi, Biot number; c, specific heat, J/(kg·K); Ki, Kirpichov number; l, dimensionless thickness of the thermal layer; m, parameters of the shape of a body; Pd, Predvoditelev number; Pk, Postol'nik number; Po, Pomerantsev number; q_s, surface specific heat flux, W/m^2; R, half the thickness of a plate, radius of a cylinder or a sphere, m; r, absolute coordinate reckoned from the center of the body's cross section, m; $T(x, y, z)$, temperature of a body at the point with coordinates x, y, z at the instant of time t, K; T_0, initial temperature of a body, K; T_s, surface temperature of a body, K; T_1, temperature of the first body, K; T_2, temperature of the second body, K; T_m, prescribed temperature of the ambient medium, K; T^*, characteristic temperature, K; t, time, sec; V_T, rate of change of the surface temperature of a body, K/sec; α, heat-transfer coefficient, W/(m^2·K); γ, density, kg/m^3; λ, thermal conductivity, W/(m·K); η, Kirchhoff number; θ, relative excess temperature; σ_{app}, apparent coefficient of radiant heat exchange, W/(m^2·K^4); τ and Fo, dimensionless time; ξ, ρ, dimensionless coordinates; ω, specific power of the internal heat sources, W/m^3. Subscripts: n, approximation No.; s, body's surface; c.s, combined surface of two bodies; m, ambient medium; c, center of a body; the first numerical index (or numerical index before the letter) i (i = 1, 2) denotes No. of the heating step (inertial i = 1 or ordered i = 2), the second numerical index j (j = 1, 2, 3, ...) denotes No. of approximation; app, apparent; max, maximum.

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